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On Johnson's Example of a Nonconvex Chebyshev Set

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The note identifies two errors in the proof of the Chebyshevity of Johnson's example of a nonconvex subset of the inner product space of all finite sequences. Corrections are given. © 1993 Academic Press, Inc.

In 1934, Bunt [B] proved that each Chebyshev set in a finite-dimensional Hilbert space must be convex. Several different proofs of this fact, along with some generalizations, were later given by numerous researchers (see, e.g., [D] for a detailed historical exposition). However, it is still unknown after 58 years whether this is true in infinite-dimensional Hilbert space. Klee [K] conjectured, and provided supporting evidence, that there exist nonconvex Chebyshev sets in infinite-dimensional Hilbert spaces.

In a recent paper, Gordon G. Johnson [J] constructed the first example of a nonconvex Chebyshev set in the incomplete inner product space of all finite sequences. We paraphrase a quote from [D]: "This example is the closest thing to a nonconvex Chebyshev set in a Hilbert space that has been constructed. It strongly supports Klee's conjecture that a nonconvex Chebyshev set must exist in some infinite-dimensional Hilbert space." Johnson's proof that the set he constructed is Chebyshev is ingenious, lengthy, and complicated. We found two flaws in his proof but were able to repair them. In fact, our corrections provide a concise proof of the Chebyshevity of the nonconvex set. We feel that Johnson's example may provide the framework for resolving the problem, and for answering Klee's conjecture. It is our hope that we have assisted the interested readers in a better understanding of Johnson's proof.

We state Johnson's example and adopt his notation without further explanation:

$$E_n = \{(x_1, x_2, ..., x_n, 0, 0, ...) : x_i \in \mathbb{R}, 1 \leqslant i \leqslant n\}$$

$$E = \bigcup_{n=1}^{\infty} E_n.$$

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For $x=(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in E$, define the norm of x by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{1/2}$. That is, E is the subspace of l^2 consisting of all elements having finitely many nonzero components.

$$a_0 = 2, \qquad A_0 = 1, \qquad F_0 = 1, \qquad L_0 = 1$$

$$d_1 = \{(x_1, 0, 0, \ldots) : -F_0 \leqslant x_1 \leqslant a_0 F_0\}$$

$$L_1(x_1) = a_0 F_0^2 + (a_0 - 1) F_0 x_1 - x_1^2, \qquad (x_1, 0, 0, \ldots) \in d_1$$

$$F_1^2(x_1) = 2L_1(x_1)/[a_0 + 1]$$

$$S_1 = \{(x_1, -F_1(x_1), 0, 0, \ldots) : (x_1, 0, 0, \ldots) \in d_1\}.$$

Inductively,

$$a_n = 1 + A_n L_n, \qquad \text{where } A_n \text{ is a positive number to be determined later.}$$

$$d_{n+1} = \{(x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots) : -F_n \leqslant x_{n+1} \leqslant a_n F_n\},$$

$$L_{n+1} = a_n F_n^2 + (a_n - 1) F_n x_{n+1} - x_{n+1}^2, \qquad (x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots) \in d_{n+1},$$

$$F_{n+1}^2 = 2L_{n+1}/[a_n + 1],$$

$$S_{n+1} = \{(x_1, x_2, \ldots, x_{n+1}, -F_{n+1}, 0, 0, \ldots) : (x_1, x_2, \ldots, x_{n+1}, 0, 0, \ldots) \in d_{n+1}\}.$$

Then, $\bigcup_{n=1}^{\infty} S_n$ is a nonconvex Chebyshev subset of E.

The proof is based on the following four statements:

STATEMENT 1. *A_n can be chosen such that any eigenvalue λ of the matrix $(D_{i,j} F_{n+1}^2/2)$ satisfies $-1 < \lambda < 0$.*

STATEMENT 2. *Suppose that K is a closed, bounded, and convex subset of $\mathbb{R}^n$ which has nonempty interior, and D is another bounded closed set in $\mathbb{R}^n$ which has a simply connected interior. If $F: K \to \mathbb{R}^n$, is continuous, $F|_{\partial K}$ is a homeomorphism from ∂K to ∂D, where ∂K, ∂D are boundaries of K and D, respectively, and F is locally a homeomorphism in the interior of K, then F is a homeomorphism from K to D.*

STATEMENT 3. *d_{n+1} is bounded, closed, and convex. Let $G_{n+1}(x_1, x_2, \ldots, x_{n+1})$ denote $((D_1 F_{n+1}^2/2) + x_1,\ (D_2 F_{n+1}^2/2) + x_2, \ldots, (D_{n+1} F_{n+1}^2/2) + x_{n+1})$. Then G_{n+1} is a homeomorphism from d_{n+1} onto its image. Let I_{n+1} be the set bounded by S_{n+1} and d_{n+1}. Then I_{n+1} is convex.*

STATEMENT 4. *$S_n \subset S_{n+1}$, for $n = 1, 2, \ldots$. Let $E_n = \{(x_1, \ldots, x_n, 0, 0, \ldots) : x_i \in \mathbb{R}\}$. Then, for any y in E_n, with respect to $\bigcup_{n=1}^{\infty} S_n$, y has a unique nearest point in S_{n+1}: $d(y, \bigcup_{n=1}^{\infty} S_n) = d(y, S_{n+1})$.*

Since $E = \bigcup_{n=1}^{\infty} E_{n+1}$, then $\bigcup_{n=1}^{\infty} S_n$ is a nonconvex Chebyshev subset of E by Statement 4. Once we have proved Statements 1 and 2, the proofs for Statements 3 and 4 are clear and can be found in Johnson's paper. Statement 2 appeared as Lemma A in [J]. The proof contained gaps and is not convincing. Also, in the proof that $|JG_{n+1}| = |D_{i,j}F_{n+1}^2/2 + \delta_{ij}| \neq 0$ on page 312 of [J], a formula for the expansion of a determinant was incorrectly applied. However, both of these errors can be rectified so that Johnson's example is correct.

We first prove Statement 2. It is a purely topological lemma. One can show that the condition that D have a simply connected interior is unnecessary. But for Johnson's example, this weaker form of the statement is sufficient.

Proof of Statement 2. Since ∂D is homeomorphic to ∂K, which is homeomorphic to the unit sphere of $\mathbb{R}^n$, then, by the Jordan Separation Theorem, $\mathbb{R}^n - \partial D = W_1 \cup W_2$, where W_1, W_2 are disjoint, open and connected sets with W_1 bounded. Suppose $F(K) \not\subset D$. Then there exists $y \notin D$, $y \in F(K)$ such that any open neighborhood of y contains some points not in $F(K)$. Since $y \in F(K)$, $y \notin D$, there exists $x \in \operatorname{int} K$ with $F(x) = y$. But this is impossible since y is not an interior point of $F(K)$, which contradicts the assumption that F is locally a homeomorphism. So, we have that $F(K) \subset D$ and $F(\operatorname{int} K) \subset \operatorname{int} D$. Moreover, note that $\operatorname{int} D \backslash F(\operatorname{int} K) = \operatorname{int} D \backslash F(K)$ is both open and closed in $\operatorname{int} D$ and that $F(\operatorname{int} K) = \operatorname{int} D$. Thus $F\colon K \to D$ is onto.

For any $y \in \operatorname{int} D$, $F^{-1}(y)$ is a compact set with the discrete topology. Thus, the cardinality of $F^{-1}(y)$ is finite. We may assume that $F^{-1}(y) = \{x_1, x_2, ..., x_k\}$. Since F maps ∂K to ∂D, $x_i \in \operatorname{int} K$ for each $1 \leqslant i \leqslant k$. Note that K is compact. Then, the assumption that F is locally a homeomorphism implies that card $F^{-1}(y)$ is locally a constant. Therefore, it has to be a constant throughout $\operatorname{int} D$, for $\operatorname{int} D$ is pathwise connected. Then F is a covering map from $\operatorname{int} K$ to $\operatorname{int} D$. Note that $\operatorname{int} D$ is simply connected. We conclude that F is one to one. Thus $F\colon K \to D$ is one to one, continuous, and onto, i.e., a homeomorphism.

In order to show Statement 1, we need three lemmas. Lemma 1 is a combination of Lemmas E and F in Johnson's paper. The proof of Lemma 2 is quite straightforward and is omitted here.

LEMMA 1. *$D_i L_n$, $D_{i,j} L_n$ are bounded on d_n for $1 \leqslant i, j \leqslant n$.*

LEMMA 2. *$G(x_1, ..., x_n, y)$ is a convex function on a convex set $D \subset \mathbb{R}^{n+1}$ if and only if for each $(\beta_1, ..., \beta_n) \in \mathbb{R}^n$ and each $(x_1, ..., x_n, y) \in D$ the function $H(t, y)$ defined by*

$$H(t, y) = G(x_1 + t\beta_1, x_2 + t\beta_2, ..., x_n + t\beta_n, y)$$

is a convex function on its domain (which is a nonempty convex subset of $\mathbb{R}^2$).

LEMMA 3. *Let* $f(t) = F(x_1 + t\beta_1, x_2 + t\beta_2, ..., x_n + t\beta_n)$, *where* $F(x_1, ..., x_n)$ *has the second order derivatives on a convex set D. Then* $-1 < f_t''(0) < 0$, *for all possible unit vectors* $(\beta_1, ..., \beta_n) \in \mathbb{R}^n$ *if and only if* $-1 < \lambda < 0$ *holds for any eigenvalue* λ *of* $(D_{i,j}F)$.

Proof. It suffices to observe that

$$f_t''(0) = \sum_{i,j=1}^{n} D_{i,j}F\beta_i\beta_j \qquad \text{at } (x_1, ..., x_n).$$

Proof of Statement 1. We prove the statement by induction on n. When $n = 1$,

$$L_1(x_1) = 2 + x_1 - x_1^2 \qquad \text{and} \qquad F_1^2(x_1)/2 = \tfrac{1}{3}(2 + x_1 - x_1^2)$$
$$D_{1,1}F_1^2/2 = -\tfrac{2}{3}.$$

Then $-1 < D_{1,1}F_1^2/2 < 0$. Now suppose $-1 < \lambda < 0$ is true for any eigenvalue λ of $(D_{i,j}F_n^2/2)$. Let $f(t) = F_n(x_1 + t\beta_1, x_2 + t\beta_2, ..., x_n + t\beta_n)$, where $(x_1, ..., x_n) \in d_n$, and $(\beta_1, ..., \beta_n)$ is a unit vector in E_n such that $(x_1 + t\beta_1, x_2 + t\beta_2, ..., x_n + t\beta_n) \in d_n$ for some t around zero. Let $a(t) = a_n(x_1 + t\beta_1, x_2 + t\beta_2, ..., x_n + t\beta_n)$.

We list some obvious facts:

1. ff_t' is bounded on d_n if and only if $F_n(D_iF_n)$ is bounded on d_n.

2. $D_ia_n = A_nD_iL_n$, $D_{i,j}a_n = A_nD_{i,j}L_n$. A_n is to be determined later. However, we may assume $0 < A_n < 1$. Then we have

$$\lim_{A_n \to 0} a'(t) = \lim_{A_n \to 0} a''(t) = \lim_{A_n \to 0} (a(t) - 1) = 0 \qquad \text{on } d_n.$$

To show that $(D_{i,j}F_{n+1}^2/2)$ is negative definite, it suffices to show that

$$(D_{t,y}H(t, y)) = \left(D_{t,y}\left(\frac{a(t)f^2(t) + (a(t) - 1)f(t)y - y^2}{a(t) + 1}\right)\right)$$

is negative definite for an appropriate A_n. Without causing any confusion, we use a, f, a', a'', f' instead of $a(t), f(t), a'(t), a''(t), f'(t)$, respectively, in the following calculation.

A straightforward calculation gives us

$$\frac{\partial H}{\partial t} = \frac{a'(f + y)^2}{(a + 1)^2} + \frac{2aff' + (a - 1)f'y}{a + 1}$$

$$\frac{\partial H}{\partial y} = \frac{(a-1)f - 2y}{a+1}, \qquad \frac{\partial^2 H}{\partial y^2} = -\frac{2}{a+1} < 0$$

$$\frac{\partial^2 H}{\partial t\, \partial y} = \frac{a'f + (a-1)f'}{a+1} - \frac{a'((a-1)f - 2y)}{(a+1)^2} = \frac{(a-1)f'}{a+1} + \frac{2a'(f+y)}{(a+1)^2}$$

and

$$\frac{\partial^2 H}{\partial t^2} = \frac{2a'ff' + 2af'^2 + 2aff'' + a'f'y + (a-1)f''y}{a+1} - \frac{a'(2aff' + (a-1)f'y)}{(a+1)^2}$$
$$+ \frac{a''(f+y)^2 + 2a'(f+y)f'}{(a+1)^2} - \frac{2a'(f+y)^2\, a'}{(a+1)^3}.$$

Then

$$\left(\frac{\partial^2 H}{\partial t\, \partial y}\right)^2 - \frac{\partial^2 H}{\partial t^2}\frac{\partial^2 H}{\partial y^2} = \frac{(a-1)^2 f'^2}{(a+1)^2} + \frac{4a'(f+y)(a-1)f'}{(a+1)^3} + \frac{4a'^2(f+y)^2}{(a+1)^4}$$
$$+ \frac{4a'ff' + 4af'^2 + 4aff'' + 2a'f'y + 2(a-1)f''y}{(a+1)^2}$$
$$+ \frac{2a''(f+y)^2 + 4a'f'(f+y) - 2a'f'(2af + (a-1)y)}{(a+1)^3}$$
$$- \frac{4a'^2(f+y)^2}{(a+1)^4}.$$

Note that $L_n f'^2$, $(f+y)f'$, $(f+y)^2$, f, y, $L_n f''y$ are all bounded on d_n and that $\lim_{A_n \to 0}(a+1) = 2$, $\lim_{A_n \to 0} a' = \lim_{A_n \to 0} a'' = \lim_{A_n \to 0}(a-1) = 0$. The main part of

$$\left(\frac{\partial^2 H}{\partial t\, \partial y}\right)^2 - \frac{\partial^2 H}{\partial t^2}\frac{\partial^2 H}{\partial y^2} \quad \text{is} \quad \frac{4af'^2 + 4aff''}{(a+1)^2}$$

when A_n is small. But $2(f'^2 + ff'') = (f^2)''_t = \sum_{i,j} D_{i,j} F_n^2 \beta_i \beta_j < 0$ by assumption. Therefore, we can choose A_n small enough that

$$\left(\frac{\partial^2 H}{\partial t\, \partial y}\right)^2 - \frac{\partial^2 H}{\partial t^2}\frac{\partial^2 H}{\partial y^2} < 0.$$

Since $\partial^2 H/\partial y^2 < 0$, we conclude that $(D_{t,y} H(t, y))$ is negative definite. To show that all eigenvalues of $(D_{i,j} F_{n+1}^2/2)$ are greater than -1, we need to show that $(D_{i,j} F_{n+1}^2/2 + \delta_{i,j})$ is positive definite, which is equivalent to showing that

$$\begin{pmatrix} 1+\dfrac{\partial^2 H}{\partial t^2} & \dfrac{\partial^2 H}{\partial t\,\partial y} \\ \dfrac{\partial^2 H}{\partial t\,\partial y} & 1+\dfrac{\partial^2 H}{\partial y^2} \end{pmatrix}$$

is positive definite.

Note that

$$\frac{\partial^2 H}{\partial y^2}+1=\frac{-2}{a+1}+1=\frac{a-1}{a+1}>0,$$

and that

$$\begin{aligned}
&\left(\frac{\partial^2 H}{\partial t\,\partial y}\right)^2-\frac{\partial^2 H}{\partial t^2}\frac{\partial^2 H}{\partial y^2}-\frac{\partial^2 H}{\partial t^2}-\frac{\partial^2 H}{\partial y^2}-1\\
&\quad=\frac{(a-1)^2 f'^2}{(a+1)^2}+\frac{4a'(f+y)(a-1)f'}{(a+1)^3}\\
&\qquad+\frac{4a'ff'+4af'^2+4aff''+2a'f'y+2(a-1)f''y}{(a+1)^2}\\
&\qquad+\frac{2a''(f+y)^2+4a'f'(f+y)-2a'f'(2af+(a-1)y)}{(a+1)^3}\\
&\qquad-\frac{2a'ff'+2af'^2+2aff''+a'f'y+(a-1)f''y}{a+1}\\
&\qquad-\frac{a''(f+y)^2+2a'f'(f+y)-a'f'(2af+(a-1)y)}{(a+1)^2}\\
&\qquad+\frac{2a'^2(f+y)^2}{(a+1)^3}-\frac{a-1}{a+1}\\
&\quad=\frac{(a-1)^2 f'^2}{(a+1)^2}+\frac{4a'(f+y)(a-1)f'}{(a+1)^3}\\
&\qquad-\frac{(a-1)(2a'ff'+a'f'y+(a-1)f''y)}{(a+1)^2}\\
&\qquad-\frac{(a-1)\,2a(f'^2+ff'')}{(a+1)^2}\\
&\qquad-\frac{(a-1)[a''(f+y)^2+2a'f'(f+y)-a'f'(2af+(a-1)y)]}{(a+1)^3}\\
&\qquad+\frac{2a'^2(f+y)^2}{(a+1)^3}-\frac{a-1}{a+1}.
\end{aligned}$$

By an argument similar to that given before, we have that the main part of the above expression is

$$-(a-1)\left(\frac{2a(f'^2+ff'')}{(a+1)^2}+\frac{1}{a+1}\right)=-\frac{a-1}{(a+1)^2}(a+1+2a(f'^2+ff'')).$$

Since $f'^2+ff''>-1$ by assumption and d_n is a compact set in E_n, we can choose A_n to be small enough that $a+1+2a(f'^2+ff'')>0$. Then

$$\left(\frac{\partial^2 H}{\partial t\,\partial y}\right)^2-\frac{\partial^2 H}{\partial t^2}\frac{\partial^2 H}{\partial y^2}-\frac{\partial^2 H}{\partial t^2}-\frac{\partial^2 H}{\partial y^2}-1<0.$$

Hence,

$$\begin{pmatrix} 1+\frac{\partial^2 H}{\partial t^2} & \frac{\partial^2 H}{\partial t\,\partial y} \\ \frac{\partial^2 H}{\partial t\,\partial y} & 1+\frac{\partial^2 H}{\partial y^2} \end{pmatrix}$$

is positive definite.

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